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Two-point Padé approximants for the expansions of Stieltjes functions in real domain

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Abstract

The fundamental inequalities for the sequences of subdiagonal and diagonal one-point Padé approximants to Stieltjes function has been extended to the case of certain two-point Padé approximants. The results can be applied to the theory of inhomogeneous media for calculating the bounds for the effective transport coefficients of two-components heterogeneous materials.

Keywords: Two-point Padé approximants; Continued fractions; Effective conductivity

AMS classification: 30-01; 30-02; 30B70; 30E05; 30E10; 78-08

1. Introduction

One of the most important results of the theory of one-point Padé approximants to Stieltjes functions defined in real domain is [1–3, 5, 7, 18]: the descending staircase sequence of one-point Padé approximants form upper and lower bounds uniformly approaching the Stieltjes function [1, Theorem 15.2], [2, Theorem 5.2.2].

On the contrary the two-point Padé approximants has not been investigated as extensively as one-point Padé ones. The theoretical results presented in the literature are concerned mostly with so-called balanced situation [8–10]. Monotone sequences of two-point Padé approximants forming upper and lower bounds for Stieltjes functions have been reported in [8, Theorem 2.3]. The convergence of two-point Padé approximants also has been investigated in [9–11]. In [6] the mathematical properties of a special type of two-point Padé approximants for Stieltjes functions (2PTA) are examined.

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The purpose of this paper is to study the convergence of a sequence of so-called unbalanced two-point Padé approximants constructed in the real domain from asymptotic expansions of Stieltjes functions.

As an example of practical applications, the bounds for the effective conductivity of a composite consisting of cylinders, arranged in a square array, have been calculated.

This paper is organized as follows: In Section 2: (a) the basic definitions of both Stieltjes functions and two-point Padé approximants to Stieltjes functions are introduced, (b) some important theoretical results for one-point Padé approximants are recalled. In Sections 3 and 4 we prove the uniform and monotone convergence of two-point Padé approximants to a Stieltjes function. In Section 5 we present the inequalities valid for both one- and two-point Padé approximants. In Sections 6 and 7 we demonstrate some practical applications. In Section 8 final remarks complete the paper.

2. Problem formulation

In this paper we deal with a Stieltjes function $f(x)$ defined in the real domain $0 < x < \infty$ by means of the following Stieltjes-integral representation:

$$f(x) = x \int_0^\infty \frac{\mathrm{d}\zeta(u)}{1 + xu}, \quad 0 < x < \infty. \tag{2.1}$$

The spectrum $\zeta(u)$ is a real, bounded, nondecreasing function on $0 \leqslant u < \infty$ such that the asymptotic expansion of $f(x)$ (2.1) at $x = 0$ satisfies the Carleman criterion:

$$f(x) \simeq \sum_{n=1}^{\infty} -c_n(-x)^n, \tag{2.2a}$$

$$\sum_{n=1}^{\infty} c_n^{-1/2n} = \infty, \tag{2.2b}$$

where the coefficients

$$c_n = \int_0^\infty u^{n-1}\,\mathrm{d}\zeta(u) \tag{2.3}$$

appearing in (2.2) are real and finite. The Carleman condition (2.2b) is sufficient for the convergence of a sequence of one-point Padé approximants to Stieltjes function (2.1) [2, Theorem 5.5.1]. The asymptotic expansion of $f(x)$ (2.1) at $x = \infty$ takes the form

$$f(x) \simeq C_1 x + \sum_{n=0}^{\infty} C_{-n}(-x)^{-n}, \tag{2.4}$$

where C_{-n}, $n = -1, 0, 1, \ldots$ are finite. For the case of $C_1 > 0$ the coefficients C_{-n} are not moments of the spectrum $\zeta(u)$. Two-point Padé approximants calculated for power series (2.2) and (2.4) have the following general form:

$$[M+1/M]_k = \frac{L_{k,M+1}(x)}{Q_{k,M}(x)} = \frac{a_{1,k}x + a_{2,k}x^2 + \cdots + a_{M+1,k}x^{M+1}}{1 + b_{1,k}x + b_{2,k}x^2 + \cdots + b_{M,k}x^M}. \tag{2.5}$$

Consider the power expansion of (2.5) at $x = 0$

$$[M + 1/M]_k = \sum_{n=1}^{\infty} -c_{n,k}(-x)^n \tag{2.6}$$

and at $x = \infty$

$$[M + 1/M]_k = C_{1,k}x + \sum_{n=0}^{\infty} C_{-n,k}(-x)^{-n}. \tag{2.7}$$

By definition, the rational function (2.5) is the two-point Padé approximant $[M + 1/M]_k$ to the Stieltjes function (2.1), if

$$c_{n,k} = c_n \quad \text{for } n = 1,2,\ldots,2M + 1 - k \tag{2.8}$$

and

$$C_{-n,k} = C_{-n} \quad \text{for } n = -1,0,1,\ldots,k-2. \tag{2.9}$$

According to the above definition $[M + 1/M]_0$ denotes the one-point Padé approximant. Two-point Padé approximants (2.5) can also be expressed in a form of S-continued fractions [1]

$$[M + 1/M]_k = \frac{g_{1,k}x}{1} \; + \cdots + \; \frac{G_{2M+2-k,k}x}{1} \; + \cdots + \; \frac{G_{2M+1,k}x}{1}, \tag{2.10}$$

or alternatively

$$[M + 1/M]_k = \frac{g_{1,k}}{s} \; + \cdots + \; \frac{G_{2M+2-k,k}}{1} \; + \cdots + \; \frac{G_{2M+1,k}}{s}, \tag{2.11}$$

where

$$s = 1/x. \tag{2.12}$$

The coefficients $g_{l,k}, \ldots, g_{2M+1-k,k}$ of the continued fractions (2.10) and (2.11) are uniquely determined by the $2M + 1 - k$ coefficients c_n ($n = 1,2,\ldots,2M + 1 - k$) of a Stieltjes series (2.2). To determine the remaining coefficients $G_{2M+2-k,k}, \ldots, G_{2M+1,k}$ the values of k coefficients of a series (2.4) C_{-n} ($n = -1,0,1,2,\ldots,k-2$) are additionally needed.

Let us consider the case $C_1 \to 0^+$. Two-point Padé approximants calculated for series (2.2)–(2.4) take then the following general form:

$$[M/M]_k = \frac{L_{k,M}(x)}{Q_{k,M}(x)} = \frac{a'_{1,k}x + a'_{2,k}x^2 + \cdots + a'_{M,k}x^M}{1 + b'_{1,k}x + b'_{2,k}x^2 + \cdots + b'_{M,k}x^M}. \tag{2.13}$$

The rational function (2.13) is the two-point Padé approximant $[M/M]_k$ to the series (2.2)–(2.4), if

$$c_{n,k} = c_n \quad \text{for } n = 1,2,\ldots,2M - k \tag{2.14}$$

and

$$C_{-n,k} = C_{-n} \quad \text{for } n = 0,1,2,\ldots,k-1. \tag{2.15}$$

According to the above definition $[M/M]_0$ denotes the one-point Padé approximant. It is worth noting that due to Definition (2.5) and (2.13):

$$\text{for } C_1 \to 0^+, \quad a_{M+1,k}X^{M+1} \to 0, \quad [M + 1/M]_{k+1} \to [M/M]_k. \tag{2.16}$$

Thus we can write

$$[M+1/M]_{k+1} = [M/M]_k \quad \text{for } C_1 = 0. \tag{2.17}$$

We start our discussion by recalling some most important results for one-point Padé approximants $[M+J/M]_0$ ($J=0,1$) to a Stieltjes function [1–3]:

(1) $[M+J/M]_0$ has the continued fraction representation of type S

$$[M+J/M]_0 = \frac{g_{1,0}}{s} \, \frac{}{+} \, \frac{g_{2,0}}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{g_{2M+j,0}}{\alpha}, \quad \alpha = \begin{cases} 1, & J=0, \\ s, & J=1. \end{cases} \tag{2.18}$$

(2) The coefficients of the continued fraction (2.18) are positive

$$g_{n,0} > 0. \tag{2.19}$$

(3) For x real and positive the Padé approximants $[M+J/M]_0$ obey the following fundamental inequalities [1, Theorem 15.2]:

$$(-1)^J [M+J/M]_0 < (-1)^J [M+J+1/M+1]_0 < (-1)^J f(x), \quad J=0,1. \tag{2.20}$$

(4) The sequence of $[M+J/M]_0$ converges uniformly to $f(x)$ (2.1) in compact subsets of $(0,\infty)$:

$$\lim_{M\to\infty} [M+1/M] = \lim_{M\to\infty} [M/M]_0 = f(x). \tag{2.21}$$

The main purpose of this paper is to derive inequalities analogous to (2.20) valid for two-point Padé approximants $[M+1/M]_1$, $[M+1/M]_2$.

3. Inequalities for Padé approximants $[M+1/M]_1$

We shall demonstrate

Lemma 1. *The sequence of two-point Padé approximants $[M+1/M]_1$ to power series (2.2) and (2.4) converges uniformly to the Stieltjes function $f(x)$ (2.1) in compact subsets of $(0,\infty)$, as M goes to infinity.*

Proof. By substituting $k=1$ into (2.11) we obtain the continued fraction representation of $[M+1/M]_1$

$$[M+1/M]_1 = \frac{g_{1,1}}{s} \, \frac{}{+} \, \frac{g_{2,1}}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{g_{2M,1}}{1} \, \frac{}{+} \, \frac{G_{2M+1,1}}{s}. \tag{3.1}$$

We now notice that

$$g_{n,0} = g_{n,1}, \quad n = 1,2,\ldots,2M, \tag{3.2}$$

which results from the fact that both $g_{n,0}$ and $g_{n,1}$ ($n=1,2,\ldots,2M$) are determined by the same set of $2M$ power expansion coefficients in (2.2). The last coefficient $G_{2M+1,1}$ appearing in (3.1) satisfies the equation

$$\lim_{s\to 0^+} s\left(\frac{g_{1,0}}{s} \, + \, \frac{g_{2,0}}{1} \, + \, \cdots \, + \, \frac{g_{2M,0}}{1} \, + \, \frac{G_{2M+1,1}}{s}\right) = C_1, \tag{3.3}$$

where C_1 denotes the first coefficient of the series (2.4). From (2.20) it holds that for all $x > 0$, all M: $[M+1/M]_0 > f(x) > [M/M]_0$. Hence also

$$\lim_{s\to 0^+} s\,[M+1/M]_0 > C_1 > \lim_{s\to 0^+} s\,[M/M]_0. \tag{3.4}$$

It is convenient to rewrite the inequality (3.4) in terms of the continued fractions of type S

$$\lim_{s\to 0^+} s\left(\cdots \, + \, \frac{g_{2M,0}}{1} \, + \, \frac{g_{2M+1,0}}{s}\right) > C_1 > \lim_{s\to 0^+} s\left(\cdots \, + \, \frac{g_{2M,0}}{1} \, + \frac{0}{s}\right) \tag{3.5}$$

where in the right-hand side of (3.5) the vanishing term $0/s$ is introduced for convenience. By substituting (3.3) into (3.5) and comparing corresponding terms in all continued fractions we conclude that

$$0 < G_{2M+1,1} < g_{2M+1,0}. \tag{3.6}$$

By using the recurrence relation for continued fraction (2.18) it can readily be proved that the following inequality is satisfied for $g_{k,0} > 0$ $(k = 1, 2, \ldots, 2M+1)$ and $s > 0$

$$\frac{\partial}{\partial g_{2M+1,0}}[M+1/M]_0 > 0. \tag{3.7}$$

Due to (3.6) and (3.7) the validity of relation (3.5) can be extended for all positive values of s.

$$[M+1/M]_0 > [M+1/M]_1 > [M/M]_0. \tag{3.8}$$

The sequences of one-point Padé approximants $[M+1/M]_0$ and $[M+1/M]_0$ converge uniformly to the function $f(x)$ (2.1) in compact subsets of $(0, \infty)$. Hence we can write

$$\lim_{M\to\infty} [M+1/M]_1 = f(x). \qquad \square. \tag{3.9}$$

Theorem 2. *The two-point Padé approximants* $[M+1/M]_1$ *to power series* (2.2) *and* (2.4) *form a monotone sequence of lower bounds converging to* $f(x)$ (2.1) *in compact subsets of* $(0, \infty)$, *as M goes to infinity.*

Proof. The difference of two successive approximants $[M+1/M]_1$ can be expressed as follows:

$$[M+2/M+1]_1 - [M+1/M]_1 = \frac{W_1(M+2/M+1)x^{2M+1}}{Q_{1,M}(x)\cdot Q_{1,M+1}(x)}, \tag{3.10}$$

where the coefficient $W_1(M+2/M+1)$ does not depend on x and $Q_{1,M}(x)$ is the denominator of $[M+1/M]_1$ in (2.5) for $k = 1$. Equation (3.10) is a consequence of the definition of two-point Padé approximants (2.5). Dividing both sides of (3.10) by x^{2M+1} and setting $x = 0$ we get

$$c_{2M+1} - c_{2M+1,1} = W_1(M+2/M+1), \tag{3.11}$$

where c_{2M+1} and $c_{2M+1,1}$ are the power series coefficients in (2.2) and (2.6) respectively. On the other hand the following equality holds:

$$\lim_{x \to 0^+} \frac{[M+1/M]_0 - [M+1/M]_1}{x^{2M+1}} = c_{2M+1} - c_{2M+1,1}. \tag{3.12}$$

Therefore

$$W_1(M+2/M+1) = \lim_{x \to 0^+} \frac{[M+1/M]_0 - [M+1/M]_1}{x^{2M+1}}. \tag{3.13}$$

It follows from (3.8) that

$$[M+1/M]_0 - [M+1/M]_1 > 0 \tag{3.14}$$

and thus

$$W_1(M+2/M+1) > 0. \tag{3.15}$$

On account of (2.5), (2.19), (3.1) and (3.6) the terms $Q_{1,M}(x)$ and $Q_{1,M+1}(x)$ in (3.10) take only positive values. From (3.10) and (3.15) we have

$$[M+2/M+1]_1 - [M+1/M]_1 > 0 \tag{3.16}$$

and by applying Lemma 1 we obtain

$$f(x) > [M+2/M+1]_1 > [M+1/M]_1. \tag{3.17}$$

The formula (3.17) expresses the monotone convergence of two-point Padé approximants $[M+1/M]_1$ to the Stieltjes function $f(x)$ (2.1) in compact subsets of $(0, \infty)$. □

4. Inequalities for two-point Padé approximants $[M+1/M]_2$

Lemma 3. *The sequence of two-point Padé approximants* $[M+1/M]_2$ *to power series* (2.2) *and* (2.4) *converges uniformly to the function* $f(x)$ (2.1) *in compact subsets of* $(0, \infty)$, *as* M *goes to infinity.*

Proof. By setting $k = 2$ in (2.11) we obtain

$$[M+1/M]_2 = \frac{g_{1,2}}{s} \; + \; \cdots \; + \; \frac{g_{2M-1,2}}{s} \; + \; \frac{G_{2M,2}}{1} \; + \; \frac{\rho_2 G_{2M,2}}{s}, \tag{4.1}$$

where we introduced the notation

$$\rho_2 = \frac{G_{2M+1,2}}{G_{2M,2}}. \tag{4.2}$$

The first $2M-1$ coefficients of the continued fractions $[M+1/M]_1$ in (3.1) and $[M+1/M]_2$ in (4.1) are identical

$$g_{n,1} = g_{n,2}, \quad n = 1, 2, \ldots, 2M-1, \tag{4.3}$$

because they are both determined by the set of $2M - 1$ power series coefficients in (2.2). The remaining coefficients ρ_2 and $G_{2M,2}$ are determined by the following two relationships:

$$\lim_{s \to 0^+} s\,[M + 1/M]_2 = C_1, \tag{4.4a}$$

$$\lim_{s \to 0^+} \left([M + 1/M]_2 - \frac{C_1}{s} \right) = C_0. \tag{4.4b}$$

where C_1 and C_0 are the series coefficients appearing in (2.4). It is convenient to rewrite the relation (3.1) as follows:

$$[M + 1/M]_1 = \frac{g_{1,1}}{s} \, \frac{}{+} \, \frac{g_{2,1}}{1} \, \frac{}{+} \, \cdots \, \frac{}{+} \, \frac{g_{2M,1}}{1} \, \frac{}{+} \, \frac{\rho_1 g_{2M,1}}{s}, \tag{4.5}$$

where

$$\rho_1 = \frac{G_{2M+1,1}}{g_{2M,1}} > 0. \tag{4.6}$$

By multiplying $[M + 1/M]_1$ (4.5) and $[M + 1/M]_2$ (4.1) by s we obtain due to (3.3) and (4.4a)

$$\lim_{s \to 0^+} \left(\frac{G_{2M,2}}{s} \, \frac{}{+} \, \frac{\rho_2 G_{2M,2}}{1} \right) = \lim_{s \to 0^+} \left(\frac{g_{2M,1}}{s} \, \frac{}{+} \, \frac{\rho_1 g_{2M,1}}{1} \right) \tag{4.7}$$

The relations (4.6), (4.7) and (2.19) yield

$$\rho_2 = \rho_1 = \rho = \frac{G_{2M+1,1}}{g_{2M,1}} > 0. \tag{4.8}$$

Two-point Padé approximants (4.1) with $\rho_2 = \rho$ (4.8) satisfy the relation (4.4b) identically for any value of $G_{2M,2} > 0$. It is convenient for further investigations to put $G_{2M,2} = \varepsilon$, $\varepsilon > 0$. We notice that

$$\lim_{\varepsilon \to 0^+} \left(\frac{s g_{1,1}}{s} \, \frac{}{+} \, \cdots \, \frac{}{+} \, \frac{g_{2M-1,1}}{s} \, \frac{}{+} \, \frac{\varepsilon}{1} \, \frac{}{+} \, \frac{\rho \cdot \varepsilon}{s} \right) = \begin{cases} s[M/M-1]_0, & s > 0, \quad (4.9a) \\ C_1, & s = 0, \quad (4.9b) \end{cases}$$

$$\frac{\partial}{\partial \varepsilon} \left(\frac{g_{1,1}}{s} \, \frac{}{+} \, \cdots \, \frac{}{+} \, \frac{g_{2M-1,1}}{s} \, \frac{}{+} \, \frac{\varepsilon}{1} \, \frac{}{+} \, \frac{\rho \cdot \varepsilon}{s} \right) < 0 \quad \text{for } \varepsilon > 0,\ s > 0, \tag{4.10}$$

$$\frac{\partial}{\partial \varepsilon} \lim_{s \to 0^+} \left(\frac{s g_{1,1}}{s} \, \frac{}{+} \, \cdots \, \frac{}{+} \, \frac{g_{2M-1,1}}{s} \, \frac{}{+} \, \frac{\varepsilon}{1} \, \frac{}{+} \, \frac{\rho \cdot \varepsilon}{s} \right) = \frac{\partial}{\partial \varepsilon} C_1 = 0 \quad \text{for } \varepsilon > 0, \tag{4.11}$$

$$\lim_{s \to 0^+} \frac{\partial}{\partial \varepsilon} \left(\frac{g_{1,1}}{s} \, \frac{}{+} \, \cdots \, \frac{}{+} \, \frac{g_{2M-1,1}}{s} \, \frac{}{+} \, \frac{\varepsilon}{1} \, \frac{}{+} \, \frac{\rho \cdot \varepsilon}{s} - \frac{C_1}{s} \right) < 0 \quad \text{for } \varepsilon \geqslant 0. \tag{4.12}$$

The rational functions $[M/M-1]_0$ given by (4.9a) are the one-point Padé approximants to the Stieltjes function $f(x)$ (2.1). Inequality (4.10) results from the recurrence relation for the S-continued fraction enclosed within the bracket in (4.10). Expression (4.11) is a consequence of (4.4a) and (4.8).

Inequality (4.12) results directly from (4.10)–(4.11). Consider the following three relations:

$$C_0 < \lim_{s\to 0^+} \lim_{\varepsilon\to 0^+} \left(\frac{g_{1,1}}{s} \,+\, \cdots \,+\, \frac{g_{2M-1,1}}{s} \,+\, \frac{\varepsilon}{1} \,+\, \frac{\rho\cdot\varepsilon}{s} - \frac{C_1}{s}\right), \tag{4.13}$$

$$C_0 = \lim_{s\to 0^+} \left(\frac{g_{1,1}}{s} \,+\, \cdots \,+\, \frac{g_{2M-1,1}}{s} \,+\, \frac{G_{2M,2}}{1} \,+\, \frac{\rho\cdot G_{2M,2}}{s} - \frac{C_1}{s}\right), \tag{4.14}$$

$$C_0 > \lim_{s\to 0^+} \left(\frac{g_{1,1}}{s} \,+\, \cdots \,+\, \frac{g_{2M-1,1}}{s} \,+\, \frac{g_{2M,1}}{1} \,+\, \frac{\rho\cdot g_{2M,1}}{s} - \frac{C_1}{s}\right). \tag{4.15}$$

Inequality (4.13) is a consequence of (4.9) and (2.20) (for all $x > 0$, and all M: $[M/M-1]_0 > f(x)$) and of (2.1) and (2.4) (for $s \to 0^+$, $f(s) - C_1/s \to C_0$). The equality (4.14) results immediately from (4.4a), (4.1) and (4.8), whereas the inequality (4.15) from (3.17), (4.5) and (4.8). Relations (4.13)–(4.15) are more readable if the continued fractions enclosed in the brackets are expanded in power series around $s = 0$. By comparing the corresponding terms of the continued fractions of (4.13)–(4.15) and taking into account (4.12) we conclude

$$0 < G_{2M-1,2} < g_{2M-1,1}. \tag{4.16}$$

Due to (4.10) and (4.16) the inequalities (4.13)–(4.15) can be extended on the half-line $s > 0$:

$$[M-1/M-2]_0 > [M+1/M]_2 > [M+1/M]_1. \tag{4.17}$$

The approximants $[M-2/M-2]_0$ and $[M/M]_1$ converge uniformly to the function $f(x)$ (2.1) in compact subsets of $(0, \infty)$, hence

$$\lim_{M\to\infty} [M+1/M]_2 = f(x). \quad \square \tag{4.18}$$

Theorem 4. *The two-point Padé approximants* $[M+1/M]_2$ *to power series* (2.2) *and* (2.4) *form a monotone sequence of upper bounds converging to the Stieltjes function* $f(x)$ (2.1) *in compact subsets of* $(0, \infty)$, *as* M *goes to infinity.*

Proof. The difference of the two successive two-point Padé approximants $[M+1/M]_2$ has the following general form:

$$[M+2/M+1]_2 - [M+1/M]_2 = \frac{W_2(M+2/M+1)x^{2M}}{Q_{1,M}(x)\cdot Q_{1,M+1}(x)}. \tag{4.19}$$

The formula (4.19) is a consequence of the definition of the two-point Padé approximants (2.5). By repeating the procedure presented in Section 3 we get the relation

$$W_2(M+2/M+1) = \lim_{x\to 0^+} \frac{[M+1/M]_1 - [M+1/M]_2}{x^{2M}} \tag{4.20}$$

analogous to (3.13). Moreover due to (4.17) the following inequality holds:

$$[M+1/M]_1 - [M+1/M]_2 < 0. \tag{4.21}$$

From (4.20) and (4.21) we have

$$W_2(M+2/M+1)<0. \tag{4.22}$$

On account of (2.5), (2.19), (4.1) and (4.16), the polynomials $Q_{2,M}(x)$ and $Q_{2,M+1}(x)$ in (4.19) take only positive values. Consequently from (4.19) and (4.22) we get

$$[M+2/M+1]_2-[M+1/M]_2<0 \tag{4.23}$$

and finally on account of (4.18) and (4.23) we obtain

$$[M+1/M]_2>[M+2/M+1]_2>f(x). \tag{4.24}$$

The last relation expresses the monotone convergence of two-point Padé approximants to the Stieltjes function (2.1). □

5. General inequalities

By analyzing the inequalities (2.20), (3.17) and (4.24) we come to the following conclusions:

Theorem 5. *The two-point Padé approximants* $[M+1/M]_k$, $k=0,1,2$ *to asymptotic expansions* $\sum_{n=1}^{\infty}-c_n(-x)^n$ *and* $C_1x+\sum_{n=0}^{\infty}C_{-n}(-x)^{-n}$ *given by* (2.2), (2.3) *and* (2.4) *with* $C_1>0$, *respectively, obey the inequalities*

$$(-1)^k[M+1/M]_k>(-1)^k[M+2/M+1]_k>(-1)^kf(x),\quad k=0,1,2, \tag{5.1}$$

where the Stieltjes function $f(x)$ (2.1) *stands for the limit as M goes to infinity of* $[M+1/M]_k$ *and x is real and positive.*

For $C_1\to0^+$ and $k=0,1$ the inequalities (5.1) take, due to the relation (2.17), the following form:

$$(-1)^{k+1}[M/M]_k>(-1)^{k+1}[M+1/M+1]_k>(-1)^{k+1}f(x). \tag{5.2}$$

Formula (5.2) agrees with inequalities for two-point Padé approximants $[M/M]_k$, $k=0,1,2$, derived directly for $C_1=0$ [16] with the aid of the procedure analogous to that presented in Sections 3 and 4. Hence we have:

Theorem 6. *The two-point Padé approximants* $[M/M]_k$ $(k=0,1,2)$ *to asymptotic expansions* $\sum_{n=1}^{\infty}-c_n(-x)^n$ *and* $\sum_{n=0}^{\infty}C_{-n}(-x)^{-n}$ *given by* (2.2), (2.3) *and* (2.4) *with* $C_1=0$, *respectively, obey the inequalities*

$$(-1)^{k+1}[M/M]_k>(-1)^{k+1}[M+1/M+1]_k>(-1)^{k+1}f(x), \tag{5.3}$$

where the Stieltjes function $f(x)$ (2.1) *stands for the limit, as M goes to infinity of* $[M/M]_k$ *and x is real and positive.*

The formulae (5.1) and (5.3) extend the one-point Padé approximants inequalities ($k=0$) for the case of two-point Padé approximants ($k=1,2$). Theorems 3 and 4 provide upper and lower bounds for Stieltjes function with respect to the given numbers of coefficients of asymptotic expansions of $f(x)$ (2.1).

It is interesting to notice that by putting $k = M$ in (5.3) we obtain the inequalities derived by González-Vera and Njåstad [8, Theorem 2.3] for balanced situation. Numerical results illustrating the practical application of Theorem 3 are presented in Sections 6 and 7.

6. Numerical example

To illustrate the inequalities (5.1) we consider the following spectrum $\zeta(u)$:

$$\zeta(u) = 0.02\, H(u) + \frac{1}{\ln 1000}\cdot \begin{cases} 0, & 0 \leqslant u \leqslant 1, \\ u - 1, & 1 < u < 1000, \\ 999, & 1000 \leqslant u \leqslant \infty, \end{cases} \tag{6.1}$$

leading to the following Stieltjes function $f(x)$

$$f(x) = 0.02x + \frac{1}{\ln 1000}\cdot \ln \frac{1 + 1000x}{1 + x}, \tag{6.2}$$

where $H(u)$ denotes the Heaviside function. The two Stieltjes power series expanded at zero

$$f(x) = 0.02x + \frac{1}{\ln 1000}\cdot \sum_{n=2}^{\infty} \frac{1}{n}(1 - 1000^n)(-x)^n \tag{6.3}$$

and at infinity

$$f(x) = 0.02x + 1 + \frac{1}{\ln 1000}\cdot \sum_{n=1}^{\infty} \frac{1}{n}(1 - 0.001^n)(-x)^{-n} \tag{6.4}$$

result immediately from (6.2). The sequences of two-point Padé approximants

$$[3/2]_k, [5/4]_k, \ldots, [17/16]_k, \quad k = 0, 1, 2, \tag{6.5}$$

have been calculated by means of the convergents of S-continued fraction (2.18), (3.1) and (4.1). The monotone convergence of one- and two-point Padé approximants $[M + 1/M]_0$, $[M + 1/M]_1$, $[M + 1/M]_2$ to Stieltjes function (6.2) is shown in Fig. 1.

7. Application to the theory of inhomogeneous media

In the linear theory of inhomogeneous media it has been proved that effective transport coefficients $\lambda_e(x)$ of composite consisting of two components of physical properties λ_1 (matrix) and λ_2 (inclusion) have Stieltjes-integral representation expressed by (2.1), where $x = (\lambda_2 - \lambda_1)/\lambda_1$ for $\lambda_2 > \lambda_1$ or $x = (\lambda_1 - \lambda_2)/\lambda_2$ for $\lambda_1 > \lambda_2$ [4]. Hence the Theorems 3 and 4 can be directly applied to the theory of inhomogeneous media to estimate the effective properties of two-components composites.

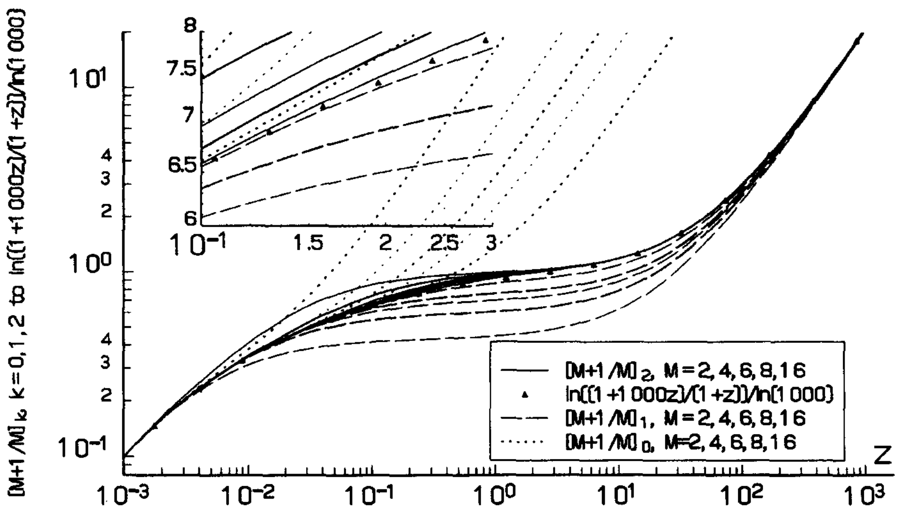

Fig. 1. The monotone sequences of one- $[M + 1/M]_0$ and two-point Padé approximants $[M + 1/M]_1$, $[M + 1/M]_2$ uniformly converging to the Stieltjes function.

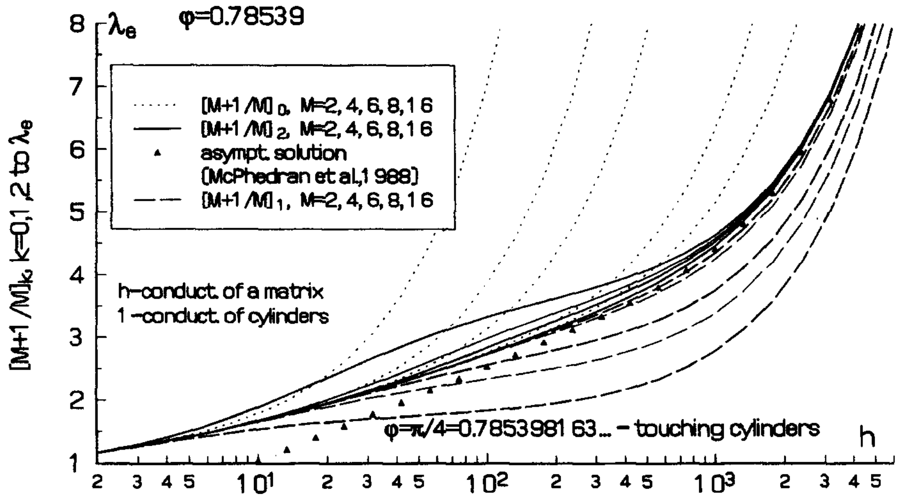

Fig. 2. The monotone sequences of Padé approximants $[M + 1/M]_0$, $[M + 1/M]_1$, $[M/M]_2$ uniformly converging to the effective conductivity λ_e of the square array of cylinders for volume fraction $\varphi = 0.78539$.

As an example, a heterogeneous material consisting of equal-sized cylinders of volume fraction φ, arranged in a square array, has been examined for $\lambda_1 > \lambda_2$. The monotone sequences of two-point Padé approximants $[M + 1/M]_0$, $[M + 1/M]_1$, $[M + 1/M]_2$ converging to the effective conductivity $\lambda_e(x)$ has been calculated by the method presented in [17]. The results are shown in Figs. 2 and 4. The best bounds $[17/16]_1$ and $[17/16]_2$, are presented in Fig. 3. In Figs. 2–4 the asymptotic solution obtained in [15] is drawn for comparison.

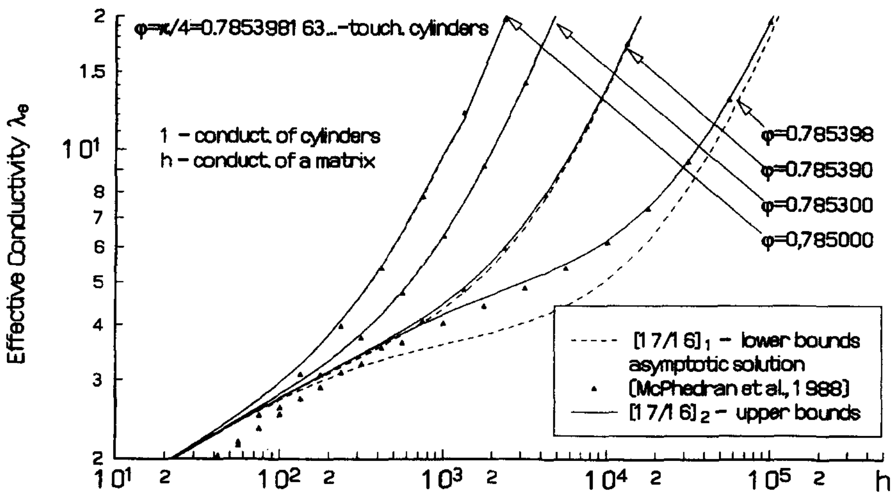

Fig. 3. The upper and lower bounds on the effective conductivity for square array of densely packed cylinders. For volume fraction $\varphi = 0.785$ the bounds coincide. For $\varphi = 0.7853, 0.78539$ are very restrictive. For higher volume fractions $\varphi \geqslant 0.78539816$ the difference between upper and lower bounds $[17/16]_2$ and $[17/16]_1$ increases rapidly.

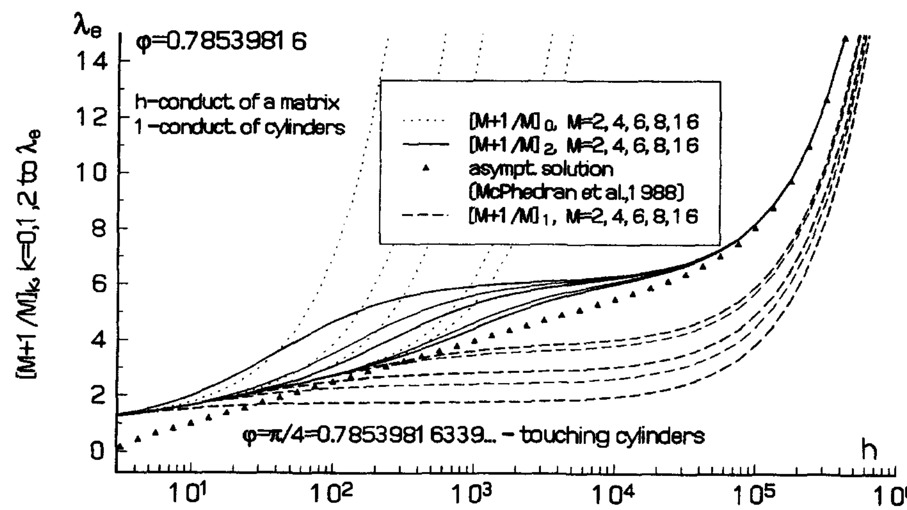

Fig. 4. The monotone sequences of Padé approximants $[M + 1/M]_0$, $[M + 1/M]_1$ and $[M + 1/M]_2$ converging to the effective conductivity λ_e of the square array of cylinders for volume fraction $\varphi = 0.78539816$.

It follows from Figs. 2–4 that the two-point Padé approximants method covers much wider a range of parameters φ and $h = \lambda_2/\lambda_1$ compared to the methods based on one-point Padé approximants [13, 14].

8. Final remarks

The inequalities for one-point Padé approximants [2, Theorem 5.2.2, Eqs. (2.7) and (2.8)] to Stieltjes series satisfying the Carleman condition [2, Theorem 5.5.1] have been extended on the case on two-point Padé approximants $[M + J/M]_k$, $J = 0, 1$; $k = 1, 2$ (Theorems 3 and 4).

As an example of practical applications the bounds for the effective conductivity of square array of closely spaced cylinders have been calculated.

Some additional numerical experiments carried out by the author suggest that the inequalities (5.1) and (5.3) hold, not only for $k = 0, 1, 2$, but also for arbitrary value of $k = 3, 4, \ldots$.

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